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# *Surfaces with the same Spherical Representation of their Lines of Curvature as Spherical Surfaces.*

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## INTRODUCTION.

In several memoirs\* we have studied the surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces. It is now our purpose to consider the surfaces with the same representation as spherical surfaces with a view to deriving significant theorems similar to those for *A*-surfaces,\* and also theorems which of necessity have no analogues in the theory of the latter surfaces.

After finding in §1 reduced forms of the equations of Gauss and Codazzi to be satisfied by the fundamental quantities of the surface, we derive the expressions of the latter for the surfaces parallel to a given spherical surface and note that two of them are surfaces of constant mean curvature — as found by Bonnet.† We say that all the surfaces with the same spherical representation form a group; evidently the spherical surface of unit curvature of the group determines the group, and in this sense it may be said to be *associated* with each member of it. Bonnet has shown‡ that, given one of the above surfaces, there is a unique surface of the same kind applicable to it with correspondence of the lines of curvature, and that these are the only surfaces possessing this property; on this account we call them *surfaces of Bonnet*. It is shown that the two

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\* Surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces, Amer. Journ., vol. 27, pp. 113–172 (1905); we have called them *A*-surfaces and hereafter this memoir is referred to thus: *A*. p. 113.

Surfaces analogous to Surfaces of Bianchi, Annali, vol. 12, pp. 113–143 (1905).

† Note sur une propriété de maximum relative à la sphere, Nouv. Annal. de Math., vol. 12, p. 433 (1853); also, Bianchi, Lezioni, II, p. 437.

‡ Mémoire sur la théorie des surfaces applicables sur une surface donnée, Journ. de l'Ecole Polytech., Cahier 42, p. 44 et seq.

spherical surfaces associated with a pair of applicable surfaces of Bonnet are the Hazzidakis transforms\* of one another.

Bianchi has established† an imaginary transformation of spherical surfaces which is similar to the Bäcklund transformations of pseudospherical surfaces. In § 2 we have given a generalization of this transformation making it applicable to any surface of Bonnet in somewhat the same manner that we did for  $A$ -surfaces. As in the case of the latter a theorem of permutability can be established so that the knowledge of the general transformation of a surface of Bonnet enables one to find, by algebraic processes, all the transformations of the transforms of the original surface. This is done in § 3.

By means of the theorems of permutability we find in § 4 two imaginary transformations which, when applied successively, transform a given real surface of Bonnet into a new real surface. In particular, we consider the case where the latter belongs to the same group as the original surface.

In § 5 we apply the generalized Bäcklund transformation to two applicable surfaces of Bonnet and show that the functions determining the transformations can be chosen so that after each surface has been subjected to two transformations the resulting surfaces of Bonnet are applicable.

With the aid of the functions giving the generalized Bäcklund transformations we can define a general transformation from a given surface of Bonnet to an imaginary one of the same group, as in the case of  $A$ -surfaces.‡ When such a transformation is known, we can find without quadrature another which together with the former transforms the original surface into a real surface of Bonnet of the same group. These results are obtained in § 6. In § 7 several particular solutions are found giving surfaces whose coordinates are expressed in forms similar to those which define the surfaces of Bianchi and the surfaces analogous to them, which we have considered elsewhere.§

In § 8 we show that, when one has a surface of Bonnet,  $S$ , and knows a Bäcklund transformation of it into another surface of Bonnet, then he can find by algebraic processes the unique surface of Bonnet applicable to  $S$  with correspondence of the lines of curvature.

\* Bianchi, *Lezioni*, II, p. 437.

† Bianchi, *Lezioni*, II, p. 452.

‡ Surfaces analogous to Surfaces of Bianchi, l. c. p. 116.

§ Ibid., pp. 118 et seq.

§ 1. *Transformation of Hazzidakis. Theorem of Bonnet.*

When a spherical surface  $\Sigma$  of total curvature  $+1$  is referred to its lines of curvature, the parameters can be so chosen that the linear elements of the surface and its spherical representation can be given the respective forms\*

$$ds^2 = \sinh^2 \omega \, du^2 + \cosh^2 \omega \, dv^2, \quad (1)$$

$$ds'^2 = \cosh^2 \omega \, du^2 + \sinh^2 \omega \, dv^2, \quad (2)$$

where  $\omega$  is a solution of

$$\frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} + \sinh \omega \cosh \omega = 0. \quad (3)$$

Denote by  $S$  any surface with its lines of curvature represented upon the sphere by the same lines as  $\Sigma$ , and write its linear element thus

$$ds^2 = E \, du^2 + G \, dv^2. \quad (4)$$

The second fundamental quantities have the forms†

$$D = \sqrt{E} \cosh \omega, \quad D' = 0, \quad D'' = \sqrt{G} \sinh \omega. \quad (5)$$

The Codazzi and Gauss fundamental equations‡ for  $S$  are satisfied, if  $E$  and  $G$  are such that

$$\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} = \frac{\partial \omega}{\partial u}, \quad \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} = \frac{\partial \omega}{\partial v}. \quad (6)$$

Surfaces satisfying these conditions will be called *surfaces of Bonnet*. When a system of lines on the sphere leads to a linear element of the form (2), the determination of all the surfaces of Bonnet with this representation of their lines of curvature requires the integration of an equation of Laplace.§ We shall say that these surfaces form a *group*, which evidently is signalized by the spherical surface  $\Sigma$  of the group. It is evident that all of the parallels of a surface of Bonnet are surfaces of the same kind.

On the assumption that  $E$  and  $G$  are functions of  $\omega$  alone the equations (6) reduce to forms from which it can be shown that the most general expressions for  $E$  and  $G$ , on the given hypothesis, are such that

$$\sqrt{E} = c_1 \sinh \omega + c_2 \cosh \omega, \quad \sqrt{G} = c_2 \sinh \omega + c_1 \cosh \omega. \quad (7)$$

\* Bianchi, II, p. 436.

† Ibid., I, p. 150.

‡ Ibid., I, p. 122.

§ cf. A, p. 118.

In particular, when  $c_2 = 0$ ,  $S$  is  $\Sigma$ , or homothetic to it, and when  $c_1 = 0$ ,  $S$  is a sphere concentric with the unit sphere. When

$$\begin{aligned} |c_1| &= |c_2| = R, \\ \text{we have} \quad E &= G = R^2 e^{\pm 2\omega}, \end{aligned} \tag{8}$$

where the upper or lower sign obtains according as  $c_1$  and  $c_2$  have the same or opposite signs. In like manner the radii of curvature have the expressions

$$\rho_1 = \pm R \frac{e^{\pm \omega}}{\cosh \omega}, \quad \rho_2 = R \frac{e^{\pm \omega}}{\sinh \omega}, \tag{9}$$

so that the mean curvature of the surfaces is  $\pm \frac{1}{R}$ . Moreover, it can be shown that, for every surface with constant mean curvature, the parameters of the lines of curvature can be chosen so that the fundamental quantities of the first order are of the form (8) and the principal radii are given by (9), or in inverse order; here  $\omega$  is any solution of equation (3).

Bonnet showed that surfaces of constant mean curvature are parallel to certain spherical surfaces. But it can be proved readily that all the surfaces of Bonnet with the spherical representation (2) and whose fundamental quantities of the first order are of the form (7), with  $c_1$  and  $c_2$  arbitrary, are parallel to the spherical surface  $\Sigma$  associated with them, or are homothetic of the surfaces parallel to it.\*

From the theory of applicability of surfaces we know that there is a double family of lines of the unit sphere for which the parameters can be chosen so that the linear element takes the form (1). If the linear element of surfaces with this representation of their lines of curvature be written in the form

$$ds_1^2 = E_1 du^2 + G_1 dv^2,$$

we may take for the fundamental quantities of the second order

$$D_1 = \sqrt{E_1} \sinh \omega, \quad D'_1 = 0, \quad D''_1 = \sqrt{G_1} \cosh \omega$$

and the Gauss and Codazzi equations reduce to (3) and

$$\frac{1}{\sqrt{E_1}} \frac{\partial \sqrt{G_1}}{\partial u} = \frac{\partial \omega}{\partial u}, \quad \frac{1}{\sqrt{G_1}} \frac{\partial \sqrt{E_1}}{\partial v} = \frac{\partial \omega}{\partial v}. \tag{10}$$

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\* cf. A, p. 124.

In the first place we remark that the latter equations are satisfied by

$$E_1 = \cosh^2 \omega, \quad G_1 = \sinh^2 \omega.$$

The corresponding surface is seen to be spherical; call it  $\Sigma_1$ . Bianchi has called it the *Hazzidakis transform* of  $\Sigma$ .\*

Again, on comparing (6) and (10) it is seen that a solution of equations (10) is given by

$$E_1 = E, \quad G_1 = G.$$

Hence the theorem:

*Every surface in the group of surfaces of Bonnet associated with a spherical surface  $\Sigma$  is applicable to one of the surfaces of Bonnet associated with the Hazzidakis transform of  $\Sigma$ .*

It is evident that the lines of curvature correspond on each pair of these applicable surfaces. Bonnet has shown that these are the only pairs of applicable surfaces with this property.

As in the case of the  $A$ -surfaces,† it can be shown that:

*The spherical surfaces and their parallels are the only surfaces of Bonnet which are Weingarten surfaces.*

## § 2. Generalized Bäcklund Transformations of Surfaces of Bonnet.

Consider a point  $M$  on a surface of Bonnet and in the tangent plane at this point draw a line through  $M$ , denoting by  $\theta$  the angle which it makes with the positive direction of the tangent to the curve  $v = \text{const.}$  through  $M$ . It is our purpose to consider the envelope of the plane which meets the tangent plane, under constant angle  $\sigma$ , in the line as above drawn.

Denote by  $M_1$  the point of contact of the above plane with its envelope. From  $M_1$  drop a perpendicular to the line of intersection of the two planes, and denote its length by  $\mu$ . Further, let  $\lambda$  denote the distance of the foot of this perpendicular from  $M$ .

We refer the surface to the moving rectangular axes formed by the tangents to the lines of curvature at  $M$  and the normal to the surface at this point.

The coordinates of  $M_1$  with respect to these axes are

$$\lambda \cos \theta - \mu \cos \sigma \sin \theta, \quad \lambda \sin \theta + \mu \cos \sigma \cos \theta, \quad \mu \sin \sigma. \quad (11)$$

\* Lezioni, II, p. 439.

† *A*, p. 125.

The projections upon these axes of a small displacement of  $M_1$  are found to be\*

$$\left. \begin{aligned} d(\lambda \cos \theta - \mu \cos \sigma \sin \theta) + \sqrt{E} du + \mu \sin \sigma \cosh \omega du \\ + (\lambda \sin \theta + \mu \cos \sigma \cos \theta) \left( \frac{\partial \omega}{\partial v} du - \frac{\partial \omega}{\partial u} dv \right), \\ d(\lambda \sin \theta + \mu \cos \sigma \cos \theta) + \sqrt{G} dv - (\lambda \cos \theta - \mu \cos \sigma \sin \theta) \\ \left( \frac{\partial \omega}{\partial v} du - \frac{\partial \omega}{\partial u} dv \right) + \mu \sin \sigma \sinh \omega dv, \\ \sin \sigma d\mu - \cosh \omega (\lambda \cos \theta - \mu \cos \sigma \sin \theta) du - \sinh \omega (\lambda \sin \theta \\ + \mu \cos \sigma \cos \theta) dv. \end{aligned} \right\} \quad (12)$$

The calculations which follow are more readily made, if we replace the preceding expressions by the projections of a displacement of  $M_1$  on the line of intersection of the planes (call it  $MP$ ), the line  $MQ$  perpendicular to the latter, lying in the tangent plane, and the normal to the surface. From (12) it follows that these projections are

$$\left. \begin{aligned} d\lambda - \mu \cos \sigma d\theta + \sqrt{E} \cos \theta du + \sqrt{G} \sin \theta dv + \left( \frac{\partial \omega}{\partial v} du - \frac{\partial \omega}{\partial u} dv \right) \mu \cos \sigma \\ + \mu \sin \sigma (\cos \theta \cosh \omega du + \sin \theta \sinh \omega dv), \\ \lambda d\theta + \cos \sigma d\mu - \sqrt{E} \sin \theta du + \sqrt{G} \cos \theta dv - \left( \frac{\partial \omega}{\partial v} du - \frac{\partial \omega}{\partial u} dv \right) \lambda \\ - \mu \sin \sigma (\sin \theta \cosh \omega du - \cos \theta \sinh \omega dv), \\ \sin \sigma d\mu - \cosh \omega (\lambda \cos \theta - \mu \cos \sigma \sin \theta) du - \sinh \omega (\lambda \sin \theta + \mu \cos \sigma \cos \theta). \end{aligned} \right\} \quad (13)$$

The direction-cosines of the given plane with respect to the lines  $MP$ ,  $MQ$ ,  $MN$  are evidently

$$0, \quad -\sin \sigma, \quad \cos \sigma. \quad (14)$$

Since this plane is to be tangent of the locus of the point  $M_1$ , the above functions must satisfy the following conditions:

$$\left. \begin{aligned} \lambda \sin \sigma \left( \frac{\partial \theta}{\partial u} - \frac{\partial \omega}{\partial v} \right) &= \sqrt{E} \sin \theta \sin \sigma - \lambda \cos \sigma \cosh \omega \cos \theta + \mu \sin \theta \cosh \omega, \\ \lambda \sin \sigma \left( \frac{\partial \theta}{\partial v} + \frac{\partial \omega}{\partial u} \right) &= -\sqrt{G} \cos \theta \sin \sigma - \lambda \cos \sigma \sinh \omega \sin \theta - \mu \cos \theta \sinh \omega. \end{aligned} \right\} \quad (15)$$

We shall consider first the case where the surface  $S$  is spherical, and inquire

whether equations can be satisfied when  $\lambda$  and  $\mu$  are constant, the latter being zero. In consequence of (1), equations (15) reduce for this case to

$$\left. \begin{aligned} \lambda \sin \sigma \left( \frac{\partial \theta}{\partial u} - \frac{\partial \omega}{\partial v} \right) &= \sin \sigma \sin \theta \sinh \omega - \lambda \cos \sigma \cos \theta \cosh \omega, \\ \lambda \sin \sigma \left( \frac{\partial \theta}{\partial v} + \frac{\partial \omega}{\partial u} \right) &= -\sin \sigma \cos \theta \cosh \omega - \lambda \cos \sigma \sin \theta \sinh \omega. \end{aligned} \right\} \quad (16)$$

Differentiate the first with respect to  $v$  and the second with respect to  $u$ , and subtract; since  $\omega$  is a solution of equation (3) the resulting equation is

$$\lambda^2 = -\sin^2 \sigma.$$

There is no loss of generality in replacing this by

$$\lambda = i \sin \sigma.$$

If this value be substituted in (16) and the resulting equations be differentiated with respect to  $u$  and  $v$  respectively, we have upon adding the equation

$$\frac{\partial^2 \theta}{du^2} + \frac{\partial^2 \theta}{dv^2} = \sin \theta \cos \theta.$$

If we introduce a new function  $\omega_1$  defined by\*

$$\theta = \frac{\pi}{2} + i \omega_1, \quad (17)$$

it is found that  $\omega_1$  is a solution of equation (3), and equations (16) take the form

$$\left. \begin{aligned} \sin \sigma \left( \frac{\partial \omega_1}{\partial u} + i \frac{\partial \omega}{\partial v} \right) &= -\sinh \omega \cosh \omega_1 + \cos \sigma \cosh \omega \sinh \omega_1, \\ \sin \sigma \left( i \frac{\partial \omega_1}{\partial v} + \frac{\partial \omega}{\partial u} \right) &= \cosh \omega \sinh \omega_1 - \cos \sigma \sinh \omega \cosh \omega_1. \end{aligned} \right\} \quad (18)$$

Now the expressions (13) reduce to

$$\begin{aligned} &-i \sinh \omega \sinh \omega_1 du + \cosh \omega \cosh \omega_1 dv, \\ &-\cos \sigma (\cosh \omega \sinh \omega_1 du + i \sinh \omega \cosh \omega_1 dv), \\ &-\sin \sigma (\cosh \omega \sinh \omega_1 du + i \sinh \omega \cosh \omega_1 dv), \end{aligned} \quad (13')$$

from which it follows that the linear element of the locus of  $M_1$  is

$$ds_1^2 = \sinh^2 \omega_1 du^2 + \cosh^2 \omega_1 dv^2. \quad (19)$$

In order to prove that the parametric lines on this surface,  $\Sigma_1$ , are the lines of curvature, we make use of a method followed by Darboux† under similar conditions in a study of the Bäcklund transformations of pseudospherical surfaces.

\* cf. Bianchi, Lezioni, II, p. 454.

† Leçons, III, p. 435.

From (14) and (17) it follows that the direction-cosines of the normal to  $\Sigma_1$  at  $M_1$  with respect to the original moving axes are

$$\sin \sigma \cosh \omega_1, \quad i \sin \sigma \sinh \omega_1, \quad \cos \sigma, \quad (20)$$

and consequently the coordinates of a point  $P$  on this normal at a constant distance  $a$  from  $M_1$  are

$$\sin \sigma (\sinh \omega_1 + a \cosh \omega_1), \quad i \sin \sigma (\cosh \omega_1 + a \sinh \omega_1), \quad a \cos \sigma.$$

Since  $\omega_1$  is a solution of equations (18), the projections upon the original axes of a displacement of the point  $P$  are reducible to

$$\begin{aligned} & - (\sinh \omega \sinh \omega_1 - \cos \sigma \cosh \omega \cosh \omega_1) (\sinh \omega_1 + a \cosh \omega_1) du \\ & \quad - i (\cosh \omega \sinh \omega_1 - \cos \sigma \sinh \omega \cosh \omega_1) (\cosh \omega_1 + a \sinh \omega_1) dv, \\ & - i (\sinh \omega \cosh \omega_1 - \cos \sigma \cosh \omega \sinh \omega_1) (\sinh \omega_1 + a \cosh \omega_1) du \\ & \quad + (\cosh \omega \cosh \omega_1 - \cos \sigma \sinh \omega_1 \sinh \omega) (\cosh \omega_1 + a \sinh \omega_1) dv, \\ & - \cosh \omega \sin \sigma (\sinh \omega_1 + a \cosh \omega_1) du \\ & \quad - i \sinh \omega \sin \sigma (\cosh \omega_1 + a \sinh \omega_1) dv. \end{aligned}$$

From these expressions it is readily found that the linear element of the locus of  $P$  is

$$ds^2 = (\sinh \omega_1 + a \cosh \omega_1)^2 du^2 + (\cosh \omega_1 + a \sinh \omega_1)^2 dv^2.$$

As defined this surface is parallel to the locus of  $M_1$ . Since the parametric lines form an orthogonal system on both surfaces they are the lines of curvature for these surfaces.

Since  $\omega_1$  is a solution of equation (3), the surface  $\Sigma_1$  with the linear element (19) is a spherical surface, whose spherical representation is given by

$$ds_1^2 = \cosh^2 \omega_1 du^2 + \sinh^2 \omega_1 dv^2. \quad (21)$$

As in the case of the Bäcklund transformations of pseudospherical surfaces, equations (18) can be transformed to the Riccati type, so that for a given value of  $\sigma$  the general integral contains an arbitrary constant. It is evident that these transforms, doubly-infinite in number, are imaginary.\*

We pass now to the consideration of the case where the surface  $S$  is any

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\* cf. Bianchi, *Lezioni*, II, p. 454.

surface of Bonnet, and discuss the case when  $\omega_1$  is any solution of equations (18). Equations (15) reduce to

$$\begin{aligned} i \sqrt{E} \sin \sigma &= \lambda \sinh \omega - i \mu \cosh \omega, \\ i \sqrt{G} \sin \sigma &= \lambda \cosh \omega - i \mu \sinh \omega, \end{aligned} \quad (22)$$

from which we get

$$\begin{aligned} \lambda &= i \sin \sigma (-\sqrt{E} \sinh \omega + \sqrt{G} \cosh \omega), \\ \mu &= \sin \sigma (-\sqrt{E} \cosh \omega + \sqrt{G} \sinh \omega), \end{aligned} \quad (23)$$

If these values for  $\lambda$  and  $\mu$  be substituted in (13), it is found that the projections of a displacement of  $M_1$  are

$$\begin{aligned} &-i \sinh \omega \sqrt{E_1} du + \cosh \omega \sqrt{G_1} dv, \\ &-\cos \sigma (\cosh \omega \sqrt{E_1} du + i \sinh \omega \sqrt{G_1} dv), \\ &-\sin \sigma (\cosh \omega \sqrt{E_1} du + i \sinh \omega \sqrt{G_1} dv), \end{aligned} \quad (24)$$

where we have put

$$\begin{aligned} \sqrt{E_1} &= \sin \sigma \left( \frac{\partial \sqrt{E}}{\partial u} - \sqrt{G} \frac{\partial \omega}{\partial u} \right) - \frac{i \lambda \sinh \omega_1 + \mu \cos \sigma \cosh \omega_1}{\sin \sigma}, \\ \sqrt{G_1} &= i \sin \sigma \left( \frac{\partial \sqrt{G}}{\partial v} - \sqrt{E} \frac{\partial \omega}{\partial v} \right) - \frac{i \lambda \cosh \omega_1 + \mu \cos \sigma \sinh \omega_1}{\sin \sigma}. \end{aligned} \quad (25)$$

From (24) we find that the linear element of the transform is

$$ds_1^2 = E_1 dw^2 + G_1 dv^2.$$

The tangent plane to this surface at a point  $M_1$  is evidently parallel to the tangent plane to the surface  $\Sigma_1$ , which is the spherical transform, by means of the same  $\sigma$  and  $\omega_1$  of the surface  $\Sigma$  associated with the original  $S$ ; corresponding points on  $\Sigma_1$  and  $S_1$  being the transforms of the points on  $\Sigma$  and  $S$  with the same spherical representation. Hence the spherical representation of  $S_1$  is given by (21), from which it follows that the parametric lines on  $S_1$  are its lines of curvature and consequently  $S_1$  is a surface of Bonnet. Therefore, each solution of equations (18) gives a transformation of the surfaces of Bonnet with the spherical representation (2) into a group with the representation (21).

From (23) it is seen that for a given  $\sigma$  the points  $M_1$ , on all the transforms of a surface of Bonnet, corresponding to a point  $M$  on the latter lie on an imaginary circle whose axis is the normal to  $S$  at  $M$ .

§ 3. *Theorem of Permutability.*

It is now our purpose to show that there exists for surfaces of Bonnet a theorem of permutability similar to the one which we established for  $A$ -surfaces.\* Thus, it will be shown that if a given surface  $S$  be transformed by means of  $(\omega_1, \sigma_1)$  and  $(\omega_2, \sigma_2)$  into the surfaces  $S_1$  and  $S_2$  respectively, there can be found without quadratures a function  $\omega_3$  such that  $S_1$  and  $S_2$  are transformed into the same surface  $S_3$  by means of  $(\omega_3, \sigma_2)$  and  $(\omega_3, \sigma_1)$  respectively.

Denote by  $\lambda_1, \mu_1$  the lengths determining the point  $M_1$  on  $S_1$  corresponding to  $M$  on  $S$ , and by  $\lambda_{13}, \mu_{13}$  the similar functions giving the transformation from  $M_1$  to  $M_3$ . From (23) it is seen that these functions are of the form

$$\left. \begin{aligned} \lambda_1 &= i \sin \sigma_1 (-\sqrt{E} \sinh \omega + \sqrt{G} \cosh \omega), \\ \mu_1 &= \sin \sigma_1 (-\sqrt{E} \cosh \omega + \sqrt{G} \sinh \omega), \\ \lambda_{13} &= i \sin \sigma_2 (-\sqrt{E_1} \sinh \omega_1 + \sqrt{G_1} \cosh \omega_1), \\ \mu_{13} &= \sin \sigma_2 (-\sqrt{E_1} \cosh \omega_1 + \sqrt{G_1} \sinh \omega_1). \end{aligned} \right\} (27)$$

Denote by  $\theta_3$  the angle formed with the tangent to the line  $v = \text{const.}$  through  $M_1$  by the line of intersection of the tangent planes to  $S_1$  and  $S_3$ . The projections, on the trihedron formed by the normal to  $S_1$  and the tangents to the lines of curvature at  $M_1$ , of the line  $M_1 M_3$  are

$$\mu_{13} \sin \sigma_2, \quad \lambda_{13} \cos \theta_3 - \mu_{13} \cos \sigma_2 \sin \theta_3, \quad \lambda_{13} \sin \theta_3 + \mu_{13} \cos \sigma_2 \cos \theta_3.$$

It is evident that this trihedron is parallel to the similar trihedron for the transform  $\Sigma_1$  of the spherical surface  $\Sigma$ . Hence it follows from (13') and (14) that the direction-cosines of the angles which the axes of the above trihedron make with the lines  $MP, MQ, MN$  for  $S$  are

$$\begin{array}{lll} -i \sinh \omega, & -\cos \sigma_1 \cosh \omega, & -\sin \sigma_1 \cosh \omega, \\ \cosh \omega, & -i \cos \sigma_1 \sinh \omega, & -i \sin \sigma_1 \sinh \omega, \\ 0, & -\sin \sigma_1, & \cos \sigma_1. \end{array}$$

Hence, if  $\theta_3$  be replaced by  $\frac{\pi}{2} + i\omega_3$ , the coordinates of  $M_3$  with respect to the axes  $MP, MQ, MN$  are

$$\begin{aligned} &\lambda_1 + \lambda_{13} \cosh (\omega_3 - \omega) - i \mu_{13} \cos \sigma_2 \sinh (\omega_3 - \omega), \\ &\mu_1 \cos \sigma_1 + \cos \sigma_1 [i \lambda_{13} \sinh (\omega_3 - \omega) + \mu_{13} \cos \sigma_2 \cosh (\omega_3 - \omega)] - \mu_{13} \sin \sigma_1 \sin \sigma_2, \\ &\mu_1 \sin \sigma_1 + \sin \sigma_1 [i \lambda_{13} \sinh (\omega_3 - \omega) + \mu_{13} \cos \sigma_2 \cosh (\omega_3 - \omega)] + \mu_{13} \cos \sigma_1 \sin \sigma_2. \end{aligned}$$

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\* *A*, p. 154.

From these it is readily found that the coordinates  $x_3, y_3, z_3$  of  $M_3$  with respect to the axes at  $M$  formed by the tangents to the lines of curvature and the normal are

$$\left. \begin{aligned} x_3 &= -i\lambda_1 \sinh \omega_1 - \mu_1 \cos \sigma_1 \cosh \omega_1 - (i\lambda_{13} \sinh \omega_1 + \mu_{13} \cos \sigma_1 \cos \sigma_2 \cosh \omega_1) \\ &\quad \cosh (\omega_3 - \omega) - (i\lambda_{13} \cos \sigma_1 \cosh \omega_1 + \mu_{13} \cos \sigma_2 \sinh \omega_1) \sinh (\omega_3 - \omega) \\ &\quad + \mu_{13} \sin \sigma_1 \sin \sigma_2 \cosh \omega_1, \\ y_3 &= \lambda_1 \cosh \omega_1 - i\mu_1 \cos \sigma_1 \sinh \omega_1 + (\lambda_{13} \cosh \omega_1 - i\mu_{13} \cos \sigma_1 \cos \sigma_2 \sinh \omega_1) \\ &\quad \cosh (\omega_3 - \omega) + (\lambda_{13} \cos \sigma_1 \sinh \omega_1 - i\mu_{13} \cos \sigma_2 \cosh \omega_1) \sinh (\omega_3 - \omega) \\ &\quad + i\mu_{13} \sin \sigma_1 \sin \sigma_2 \sinh \omega_1, \\ z_3 &= \mu_1 \sin \sigma_1 + \sin \sigma_1 [i\lambda_{13} \sinh (\omega_3 - \omega) + \mu_{13} \cos \sigma_2 \cosh (\omega_3 - \omega)] \\ &\quad + \mu_{13} \cos \sigma_1 \sin \sigma_2. \end{aligned} \right\} (28)$$

According to the statement of our problem, it must be shown that  $S_2$  is transformed by means of the same  $\omega_3$  and  $\sigma_1$ , instead of  $\sigma_2$ , into the surface  $S_3$  defined by (28). For the moment we denote the new transform by  $S'_3$  and its coordinates by  $x'_3, y'_3, z'_3$ . It is clear that the expressions for the latter are given by (28), if the subscripts 1 and 2 are interchanged and the subscript 13 is replaced by 23.

In order that the two surfaces coincide we must have

$$\begin{aligned} -i \sinh \omega_1 (x'_3 - x_3) + \cosh \omega_1 (y_3 - y'_3) &= 0, \\ -i \sinh \omega_2 (x'_3 - x_3) + \cosh \omega_2 (y_3 - y'_3) &= 0, \\ z'_3 &= z_3. \end{aligned}$$

By substitution the latter become

$$\left. \begin{aligned} &[\lambda_{23} \cosh (\omega_2 - \omega_1) - i\mu_{23} \cos \sigma_1 \cos \sigma_2 \sinh (\omega_2 - \omega_1) - \lambda_{13}] \cosh (\omega_3 - \omega) \\ &\quad + i[\mu_{13} \cos \sigma_2 - \mu_{23} \cos \sigma_1 \cosh (\omega_2 - \omega_1) - i\lambda_{23} \cos \sigma_2 \sinh (\omega_2 - \omega_1)] \\ &\quad \sinh (\omega_3 - \omega) = \lambda_1 - \lambda_2 \cosh (\omega_2 - \omega_1) + i\mu_2 \cos \sigma_2 \sinh (\omega_2 - \omega_1) \\ &\quad - i\mu_{23} \sin \sigma_1 \sin \sigma_2 \sinh (\omega_2 - \omega_1), \\ &[\lambda_{13} \cosh (\omega_2 - \omega_1) + i\mu_{13} \cos \sigma_1 \cos \sigma_2 \sinh (\omega_2 - \omega_1) - \lambda_{23}] \cosh (\omega_3 - \omega) \\ &\quad + i[\mu_{23} \cos \sigma_1 - \mu_{13} \cos \sigma_2 \cosh (\omega_2 - \omega_1) + i\lambda_{13} \cos \sigma_1 \sinh (\omega_2 - \omega_1)] \\ &\quad \sinh (\omega_3 - \omega) = \lambda_2 - \lambda_1 \cosh (\omega_2 - \omega_1) - i\mu_1 \cos \sigma_1 \sinh (\omega_2 - \omega_1) \\ &\quad + i\mu_{13} \sin \sigma_1 \sin \sigma_2 \sinh (\omega_2 - \omega_1), \\ &(\mu_{13} \sin \sigma_1 \cos \sigma_2 - \mu_{23} \sin \sigma_2 \cos \sigma_1) \cosh (\omega_3 - \omega) + i(\lambda_{13} \sin \sigma_1 - \lambda_{23} \sin \sigma_2) \\ &\quad \sinh (\omega_3 - \omega) = \sin \sigma_2 (\mu_2 - \mu_{13} \cos \sigma_1) - \sin \sigma_1 (\mu_1 - \mu_{23} \cos \sigma_2). \end{aligned} \right\} (29)$$

We consider first the case where  $S$  is a spherical surface; now

$$\lambda_1 = \lambda_{23} = i \sin \sigma_1, \quad \lambda_2 = \lambda_{13} = i \sin \sigma_2, \quad \mu_1 = \mu_2 = \mu_3 = \mu_4 = 0,$$

and the above equations reduce to

$$\begin{aligned} [\sin \sigma_1 \cosh (\omega_2 - \omega_1) - \sin \sigma_2] \cosh (\omega_3 - \omega) + \sin \sigma_1 \cos \sigma_2 \sinh (\omega_2 - \omega_1) \sinh (\omega_3 - \omega) \\ = \sin \sigma_1 - \sin \sigma_2 \cosh (\omega_2 - \omega_1), \\ [\sin \sigma_2 \cosh (\omega_2 - \omega_1) - \sin \sigma_1] \cosh (\omega_3 - \omega) - \sin \sigma_2 \cos \sigma_1 \sinh (\omega_2 - \omega_1) \sinh (\omega_3 - \omega) \\ = \sin \sigma_2 - \sin \sigma_1 \cosh (\omega_2 - \omega_1). \end{aligned}$$

Solving these equations for  $\cosh (\omega_3 - \omega)$  and  $\sinh (\omega_3 - \omega)$ , we get

$$\left. \begin{aligned} \cosh (\omega_3 - \omega) &= \frac{\sin \sigma_1 \sin \sigma_2 + (\cos \sigma_1 \cos \sigma_2 - 1) \cosh (\omega_2 - \omega_1)}{\sin \sigma_1 \sin \sigma_2 \cosh (\omega_2 - \omega_1) + \cos \sigma_1 \cos \sigma_2 - 1}, \\ \sinh (\omega_3 - \omega) &= \frac{(\cos \sigma_2 - \cos \sigma_1) \sinh (\omega_2 - \omega_1)}{\sin \sigma_1 \sin \sigma_2 \cosh (\omega_2 - \omega_1) + \cos \sigma_1 \cos \sigma_2 - 1}. \end{aligned} \right\} (30)$$

Since these expressions satisfy the general relation  $\cosh^2 \alpha - \sinh^2 \alpha = 1$ , they may be replaced by

$$\left. \tanh \left( \frac{\omega_3 - \omega}{2} \right) = \frac{\sin \left( \frac{\sigma_1 + \sigma_2}{2} \right)}{\sin \left( \frac{\sigma_1 - \sigma_2}{2} \right)} \tanh \left( \frac{\omega_2 - \omega_1}{2} \right). \right\} (31)$$

It remains for us to show that the function  $\omega_3$  thus given satisfies the conditions of the problem. The functions  $\omega_1$  and  $\omega_2$  must satisfy equations (18) in which  $\sigma$  has the respective values  $\sigma_1$  and  $\sigma_2$ . In like manner  $\omega_3$  must satisfy

$$\left. \begin{aligned} \sin \sigma_2 \left( \frac{\partial \omega_3}{\partial u} + i \frac{\partial \omega_1}{\partial v} \right) &= -\sinh \omega_1 \cosh \omega_3 + \cos \sigma_2 \cosh \omega_1 \sinh \omega_3, \\ \sin \sigma_2 \left( i \frac{\partial \omega_3}{\partial v} + \frac{\partial \omega_1}{\partial u} \right) &= \cosh \omega_1 \sinh \omega_3 - \cos \sigma_2 \sinh \omega_1 \cosh \omega_3; \end{aligned} \right\} (32)$$

and

$$\left. \begin{aligned} \sin \sigma_1 \left( \frac{\partial \omega_3}{\partial u} + i \frac{\partial \omega_2}{\partial v} \right) &= -\sinh \omega_2 \cosh \omega_3 + \cos \sigma_1 \cosh \omega_2 \sinh \omega_3, \\ \sin \sigma_1 \left( i \frac{\partial \omega_3}{\partial v} + \frac{\partial \omega_2}{\partial u} \right) &= \cosh \omega_2 \sinh \omega_3 - \cos \sigma_1 \sinh \omega_2 \cosh \omega_3. \end{aligned} \right\} (33)$$

It is readily found that, when  $\omega_1$  and  $\omega_2$  are any solutions whatever of equations (18), the function  $\omega_3$  given directly by (31) satisfies (32) and (33).\*

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\* cf. Bianchi, *Lezioni*, II, p. 458.

Furthermore, if the values of  $\sinh(\omega_3 - \omega)$  and  $\cosh(\omega_3 - \omega)$ , given by (30), are substituted in (29) together with the expressions (27) for  $\lambda$  and  $\mu$ , it is found that these conditions are satisfied. Hence we have this theorem:

*When two particular transformations of a surface of Bonnet are known, a transformation of the resulting surfaces can be effected by algebraic processes and in each case it gives the same surface.*

Consequently, as in the case of  $A$ -surfaces, when one knows the general transformation of a surface of Bonnet, its transforms can be transformed by algebraic processes.

#### § 4. Real Transformations.

From the preceding discussion it is clear that all the transforms of  $S$ , such as  $S_1$  and  $S_2$ , are imaginary; and, in general, the transforms of the latter are imaginary. We seek now surfaces of the latter class which are real.

Denoting by  $\bar{\omega}_1$ ,  $\bar{\sigma}_1$  the conjugate-imaginaries of  $\omega_1$ ,  $\sigma_1$ , we put

$$\omega_2 = i\pi - \bar{\omega}_1, \quad \sigma_2 = \pi - \bar{\sigma}_1. \quad (34)$$

It is found that  $\omega_2$  is a solution of equations (18) with  $\sigma$  given by (34), provided that  $\omega_1$  is a solution of these equations with  $\sigma_1$  in place of  $\sigma$ .

If we put for brevity

$$\begin{aligned} a &= -\sqrt{E} \sinh \omega + \sqrt{G} \cosh \omega, & b &= -\sqrt{E} \cosh \omega + \sqrt{G} \sinh \omega, \\ c &= -\sin \sigma_1 \sin \sigma_2 \left( \frac{\partial \sqrt{E}}{\partial u} - \sqrt{G} \frac{\partial \omega}{\partial u} \right), & d &= \sin \sigma_1 \sin \sigma_2 \left( \frac{\partial \sqrt{G}}{\partial v} - \sqrt{E} \frac{\partial \omega}{\partial v} \right), \end{aligned} \quad (35)$$

the expressions (27) may be written thus

$$\begin{aligned} \lambda_1 &= i \sin \sigma_1 a, & \mu_1 &= \sin \sigma_1 b, \\ \lambda_{13} &= i(c \sinh \omega_1 + i d \cosh \omega_1) + i \sin \sigma_2 a, \\ \mu_{13} &= c \cosh \omega_1 + i d \sinh \omega_1 + \cos \sigma_1 \sin \sigma_2 b. \end{aligned} \quad (36)$$

Since  $\sin \sigma_1$  and  $\sin \sigma_2$  are conjugate-imaginaries and the other functions in (35) pertain to  $S$ , the functions  $a$ ,  $b$ ,  $c$ ,  $d$  are real.

When the above values are substituted in the expression (28) for  $x_3$ , and we make use of (30), we get

$$x_3 = \frac{1}{D} \left\{ \begin{aligned} &-a (\cos \bar{\sigma}_1 + \cos \sigma_1) (\sin \bar{\sigma}_1 \cos \sigma_1 \sinh \bar{\omega}_1 + \sin \sigma_1 \cos \bar{\sigma}_1 \sinh \omega_1) \\ &+ b \cos \sigma_1 \cos \bar{\sigma}_1 (\cos \bar{\sigma}_1 + \cos \sigma_1) (\sin \bar{\sigma}_1 \cosh \bar{\omega}_1 + \sin \sigma_1 \cosh \omega_1) \\ &+ c [(\cos \bar{\sigma}_1 + \cos \sigma_1)^2 \cosh \omega_1 \cosh \bar{\omega}_1 - \sin \sigma_1 \sin \bar{\sigma}_1 \\ &\quad - (\cos \sigma_1 \cos \bar{\sigma}_1 + 1) \cosh (\bar{\omega}_1 + \omega_1) \\ &+ i d [(\cos \bar{\sigma}_1 + \cos \sigma_1) (\cos \bar{\sigma}_1 \sinh \omega_1 \cosh \bar{\omega}_1 - \cos \sigma_1 \sinh \bar{\omega}_1 \cosh \omega_1)] \end{aligned} \right\} \quad (37)$$

where  $D$  denotes the denominator in (30). Since the above expression for  $x_3$  is real and similar results follow for  $y_3$  and  $z_3$ , it is evident that  $S_3$  is a real surface.

For the values (34) equation (31) becomes

$$\tanh \left( \frac{\omega_3 - \omega}{2} \right) = \frac{\cos \left( \frac{\sigma_1 - \bar{\sigma}_1}{2} \right)}{\cos \left( \frac{\sigma_1 + \bar{\sigma}_1}{2} \right)} \coth \left( \frac{\omega_1 + \bar{\omega}_1}{2} \right), \quad \left. \vphantom{\frac{\cos \left( \frac{\sigma_1 - \bar{\sigma}_1}{2} \right)}} \right\} (38)$$

from which it is seen that  $\omega_3$  is real.

Returning to the general case, we remark that when  $\sigma_2 = \sigma_1$ , we get from (31)

$$\omega_3 - \omega = (2m + 1)i\pi. \quad (39)$$

Moreover, if this value of  $\omega_3$  be substituted in (32) and (33), they reduce to (18). Now the linear element of the spherical representation of  $S_3$ , namely

$$ds_3'^2 = \cosh^2 \omega_3 du^2 + \sinh^2 \omega_3 dv^2$$

reduces to (2). Hence  $S_3$  belongs to the same group as  $S$ ; it is the envelope of the plane containing the points  $M_1$ ,  $M_2$ , &c., which are the transforms of  $M$  by means of the general solution  $\omega_1$  of equations (18) in which  $\sigma = \sigma_1$ . We will consider, in particular, the case where  $S_3$  is real.

Referring to (38), we see that if  $\omega_3$ , given by (39), be a solution for any function  $\omega_1$ ,  $\sigma_1 + \bar{\sigma}_1$  is an odd multiple of  $\pi$ . Without loss of generality we may take

$$\sigma_1 + \bar{\sigma}_1 = \pi.$$

Now  $\sigma_1$  and  $\bar{\sigma}_1$  are of the form

$$\sigma_1 = \frac{\pi}{2} + i\tau, \quad \bar{\sigma}_1 = \frac{\pi}{2} - i\tau,$$

hence

$$\sin \sigma_1 = \sin \sigma_2 = \cosh \tau, \quad \cos \sigma_1 = \cos \sigma_2 = -i \sinh \tau. \quad (40)$$

For these values the expressions (28) for the projections upon the original trihedron of the length  $MM_3$  reduce to

$$c, \quad d, \quad b \sin^2 \sigma, \quad (41)$$

in consequence of (36).

With respect to axes fixed in space the direction-cosines of the tangents to the lines of curvature of  $\Sigma$ , and consequently of  $S$ , will be denoted by  $X_1$ ,  $Y_1$ ,  $Z_1$ ;  $X_2$ ,  $Y_2$ ,  $Z_2$ . Hence if we denote by  $(x, y, z)$  and  $(x', y', z')$  the coordinates,

with respect to these axes, of corresponding points on  $S$  and  $S_3$ , we have from (35), (40) and (41),

$$x' = x - \cosh^2 \tau \left[ \left( \frac{\partial \sqrt{E}}{\partial u} - \sqrt{G} \frac{\partial \omega}{\partial u} \right) X_1 - \left( \frac{\partial \sqrt{G}}{\partial v} + \sqrt{E} \frac{\partial \omega}{\partial v} \right) X_2 \right] + (\sqrt{E} \cosh \omega - \sqrt{G} \sinh \omega) X \quad (42)$$

and similar expressions for  $y'$  and  $z'$ . It is readily shown that these define a parallel to  $S$ , when the latter is a spherical surface or one of its parallels, and only in this case.

### § 5. *Bäcklund Transformations of Applicable Surfaces of Bonnet.*

We pass now to the consideration of the transformations of the surfaces of Bonnet whose spherical representation is given by (1). The associated spherical surface,  $\Sigma'$ , is the Hazzidakis transform of  $\Sigma$ , and its linear element is given by (2).

For this case the equations analogous to (16) are

$$\begin{aligned} \lambda' \sin \sigma' \left( \frac{\partial \theta'}{\partial u} - \frac{\partial \omega}{\partial v} \right) &= \sin \sigma' \sin \theta' \cosh \omega - \lambda' \cos \sigma' \sinh \omega \cos \theta', \\ \lambda' \sin \sigma' \left( \frac{\partial \theta'}{\partial v} + \frac{\partial \omega}{\partial u} \right) &= -\sin \sigma' \cos \theta' \sinh \omega - \lambda' \cos \sigma' \cosh \omega \sin \theta'. \end{aligned}$$

The conditions of integrability of these equations reduce to

$$\lambda' = i \sin \sigma'$$

and

$$\frac{\partial^2 \theta'}{\partial u^2} + \frac{\partial^2 \theta'}{\partial v^2} + \sin \theta' \cos \theta' = 0.$$

If we put

$$\theta' = \pi + i \omega'_1,$$

this becomes

$$\frac{\partial^2 \omega'_1}{\partial u^2} + \frac{\partial^2 \omega'_1}{\partial v^2} + \sinh \omega'_1 \cosh \omega'_1 = 0,$$

and the above equations are reducible to

$$\left. \begin{aligned} \sin \sigma' \left( \frac{\partial \omega'_1}{\partial u} + i \frac{\partial \omega}{\partial v} \right) &= i \sinh \omega'_1 \cosh \omega - i \cos \sigma' \cosh \omega'_1 \sinh \omega, \\ \sin \sigma' \left( i \frac{\partial \omega'_1}{\partial v} + \frac{\partial \omega}{\partial u} \right) &= -i \cosh \omega'_1 \sinh \omega + i \cos \sigma' \sinh \omega'_1 \cosh \omega. \end{aligned} \right\} \quad (43)$$

When these equations are compared with (18), it is seen that if  $\sigma'$  be given by

$$\sin \sigma' = i \tan \sigma, \quad \cos \sigma' = \sec \sigma, \quad (44)$$

the function  $\omega_1$  is a solution of equations (43). The linear element of the transform of  $\Sigma'_1$  by means of  $\omega_1$  and  $\sigma'$  is

$$ds_1'^2 = \cosh^2 \omega_1 du^2 + \sinh^2 \omega_1 dv^2$$

and the linear element of its spherical representation is

$$ds_1''^2 = \sinh^2 \omega_1 du^2 + \cosh^2 \omega_1 dv^2. \quad (45)$$

From these expressions it is seen that the new surface  $\Sigma'_1$  is the Hazzidakis transform of  $\Sigma_1$ .\*

We have seen that the surfaces of Bonnet associated with  $\Sigma$  and those associated with its Hazzidakis transform can be arranged in pairs of applicable surfaces. We shall consider the effect of the preceding transformations on such a pair,  $S$  and  $S'$ .

Let the linear element of  $S'$  be (4) and of its spherical representation (1). From (15) and (43) it is seen that, if we denote by  $\lambda'$  and  $\mu'$  the functions for  $S'$  analogous to  $\lambda$  and  $\mu$  for  $S$ , they are given by

$$\begin{aligned} \lambda' &= \tan \sigma (-\sqrt{E} \cosh \omega + \sqrt{G} \sinh \omega), \\ \mu' &= i \tan \sigma (\sqrt{E} \sinh \omega - \sqrt{G} \cosh \omega); \end{aligned} \quad (46)$$

in these expressions, as first found,  $\sin \sigma'$  has been replaced by  $i \tan \sigma$ . It is readily found that the linear element of the transform  $S'_1$  is

$$ds_1'^2 = E'_1 du^2 + G'_1 dv^2,$$

where

$$\left. \begin{aligned} \sqrt{E'_1} &= \tan \sigma \left( \frac{\partial \sqrt{E}}{\partial u} - \sqrt{G} \frac{\partial \omega}{\partial u} \right) - \frac{\lambda' \cosh \omega_1}{\tan \sigma} + i \mu' \frac{\sinh \omega_1}{\sin \sigma}, \\ \sqrt{G'_1} &= i \tan \sigma \left( \frac{\partial \sqrt{G}}{\partial v} - \sqrt{E} \frac{\partial \omega}{\partial v} \right) - \frac{\lambda' \sinh \omega_1}{\tan \sigma} + i \mu' \frac{\cosh \omega_1}{\sin \sigma}, \end{aligned} \right\} \quad (47)$$

and the spherical representation is given by (45).

A comparison of (23) and (46) shows that

$$\lambda' = \sec \sigma \cdot \mu, \quad \mu' = -\sec \sigma \cdot \lambda.$$

If these values be substituted in (47) and the result be compared with (25), it is found that

$$\sqrt{E'_1} = \sec \sigma \sqrt{E_1}, \quad \sqrt{G'_1} = \sec \sigma \sqrt{G_1}. \quad (48)$$

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\* cf. Bianchi, *Lezioni*, II, p. 469.

From this it is seen that a homothetic transformation applied to  $S'_1$  will give a surface of Bonnet applicable to  $S_1$ . Hence, if  $S$  and  $S'$  are two applicable surfaces of Bonnet, and  $S_1$  is the Bäcklund transform of  $S$  by means of  $\omega_1$  and  $\sigma$ , the Bäcklund transform of  $S'$  by means of  $\omega_1$  and  $\sigma'$ , the latter given by (44), is homothetic to the surface applicable to  $S_1$  with preservation of lines of curvature. All of these surfaces are imaginary, but we shall find real ones in consequence of the theorem of permutability.

As before, we denote by  $S_3$  the real surface, which is the transform of  $S_1$  by means of  $\omega_3$  and  $\sigma_2$ , these functions being given by (38) and (34); and we write the linear element of  $S_3$  in the form

$$ds_3^2 = E_3 du^2 + G_3 dv^2.$$

The preceding results show us that  $S'_1$  can be transformed into a surface  $S'_3$  by means of  $\omega_3$  and  $\sigma'_2$ , where  $\sigma'_2$  is defined by

$$\sin \sigma'_2 = i \tan \sigma_2, \quad \cos \sigma'_2 = \sec \sigma_2, \quad (49)$$

and  $S'_3$  has the same spherical representation as  $S_3$ . If the linear element of  $S'_3$  be written thus

$$ds_3'^2 = E'_3 du^2 + G'_3 dv^2,$$

the functions  $\sqrt{E_3}$ ,  $\sqrt{G_3}$ ;  $\sqrt{E'_3}$ ,  $\sqrt{G'_3}$  will have forms similar to (25) and (47). Since they are linear and homogeneous in  $\sqrt{E_1}$ ,  $\sqrt{G_1}$ ;  $\sqrt{E'_1}$ ,  $\sqrt{G'_1}$ , it follows from (48) that

$$\sqrt{E'_3} = \sec \sigma \sec \sigma_2 \sqrt{E_3}, \quad \sqrt{G'_3} = \sec \sigma \sec \sigma_2 \sqrt{G_3}. \quad (50)$$

From (34) it follows that in order that  $S'_3$  be real we must have

$$\sigma'_2 = \pi - \bar{\sigma}'. \quad (51)$$

In consequence of (34) equations (49) may be written

$$\sin \sigma'_2 = -i \tan \bar{\sigma}, \quad \cos \sigma'_2 = -\sec \bar{\sigma}$$

and from (44) it follows that

$$\sin \bar{\sigma}' = -i \tan \bar{\sigma}, \quad \cos \bar{\sigma}' = \sec \bar{\sigma}.$$

Comparing these two sets of equations, we see that condition (51) is satisfied.

In consequence of (34) equations (50) become

$$\sqrt{E'_3} = -\sec \sigma \sec \bar{\sigma} \sqrt{E_3}, \quad \sqrt{G'_3} = -\sec \sigma \sec \bar{\sigma} \sqrt{G_3}.$$

If we put

$$\sigma = \alpha + i\beta,$$

we find that

$$\sec \sigma \sec \bar{\sigma} = 1,$$

if

$$\sin^2 \alpha = \sinh^2 \beta, \quad (52)$$

and only in this case. Hence, given two applicable surfaces of Bonnet; by two imaginary transformations of Bäcklund we can obtain a second pair of applicable surfaces of Bonnet. Since  $\alpha$  or  $\beta$  is arbitrary and there is an arbitrary constant in the solution  $\omega_1$  of equations (18), there is a double infinity of these transformations.

#### § 6. *General Determination of Surfaces of Bonnet.*

In the tangent plane to a surface of Bonnet,  $S$ , at a point  $M$  we draw a line through the point of contact and indicate by  $\theta$  the angle which it makes with the tangent to the line of curvature  $v = \text{const.}$  At a point  $P$  of this line we draw in the tangent plane the segment  $PQ$  of the line perpendicular to  $PM$ . In the plane through  $PQ$  and normal to  $PM$  we draw a segment  $QR$  making an angle  $\sigma$  with  $QP$ . For convenience we indicate by  $p, \rho, r$  the respective lengths  $MP, PQ, QR$ . If  $\theta$  is defined by (17) and (18) the projections, on the trihedron formed by the tangents to the lines of curvature and the normal to  $S_1$  of the segment  $MR$  are

$$-[ip \sinh \omega_1 + (\rho + r \cos \sigma) \cosh \omega_1], [p \cosh \omega_1 - i(\rho + r \cos \sigma) \sinh \omega_1], r \sin \sigma. \quad (53)$$

From these it follows that the projections of a displacement of  $R$  are of the form\*

$$\left. \begin{aligned} & -d[ip \sinh \omega_1 + (\rho + r \cos \sigma) \cosh \omega_1] + \sqrt{E} du + r \sin \sigma \cosh \omega du \\ & \quad + [p \cosh \omega_1 - i(\rho + r \cos \sigma) \sinh \omega_1] \left( \frac{\partial \omega}{\partial v} du - \frac{\partial \omega}{\partial u} dv \right), \\ & d[p \cosh \omega_1 - i(\rho + r \cos \sigma) \sinh \omega_1] + \sqrt{G} dv + r \sin \sigma \sinh \omega dv \\ & \quad + [ip \sinh \omega_1 + (\rho + r \cos \sigma) \cosh \omega_1] \left( \frac{\partial \omega}{\partial v} du - \frac{\partial \omega}{\partial u} dv \right), \\ & \sin \sigma dr + \cosh \omega [ip \sinh \omega_1 + (\rho + r \cos \sigma) \cosh \omega_1] du \\ & \quad - \sinh \omega [p \cosh \omega_1 - (\rho + r \cos \sigma) \sinh \omega_1] dv. \end{aligned} \right\} \quad (54)$$

From these it is found that the necessary and sufficient condition that the locus of  $R$  be a surface of Bonnet with the same spherical representation of its lines of curvature as  $S$  is that  $p$ ,  $\rho$  and  $r$  satisfy the following equations

$$\left. \begin{aligned} \sin \sigma \cosh \omega_1 \frac{\partial p}{\partial u} - i \sin \sigma \sinh \omega_1 \frac{\partial \rho}{\partial u} - [p \sinh \omega_1 - i(\rho + r \cos \sigma) \cosh \omega_1] \sinh \omega \cosh \omega_1 &= 0, \\ i \sin \sigma \sinh \omega_1 \frac{\partial p}{\partial v} + \sin \sigma \cosh \omega_1 \frac{\partial \rho}{\partial v} + i[p \cosh \omega_1 - i(\rho + r \cos \sigma) \sinh \omega_1] \cosh \omega \sinh \omega_1 &= 0, \\ \sin \sigma \frac{\partial r}{\partial u} + [i p \sinh \omega_1 + (\rho + r \cos \sigma) \cosh \omega_1] \cosh \omega &= 0, \\ \sin \sigma \frac{\partial r}{\partial v} - [p \cosh \omega_1 - i(\rho + r \cos \sigma) \sinh \omega_1] \sinh \omega &= 0. \end{aligned} \right\} \quad (55)$$

From (54) one finds that the coefficients of the linear element of the new surface are given by

$$\left. \begin{aligned} \sqrt{E'} &= \sqrt{E} - \left( i \sinh \omega_1 \frac{\partial p}{\partial u} + \cosh \omega_1 \frac{\partial \rho}{\partial u} \right) + \frac{(\rho \cos \sigma + r) \cosh \omega}{\sin \sigma} \\ &\quad + \frac{i p \sinh \omega \cosh^2 \omega_1 + (\rho + r \cos \sigma) \sinh \omega_1 \cosh \omega_1 \sinh \omega}{\sin \sigma}, \\ \sqrt{G'} &= \sqrt{G} + \left( \cosh \omega_1 \frac{\partial p}{\partial v} - i \sinh \omega_1 \frac{\partial \rho}{\partial v} \right) + \frac{i(\rho \cos \sigma + r) \sinh \omega}{\sin \sigma} \\ &\quad + \frac{p \sinh^2 \omega_1 \cosh \omega - (\rho + r \cos \sigma) i \sinh \omega_1 \cosh \omega_1 \cosh \omega}{\sin \sigma}. \end{aligned} \right\} \quad (56)$$

As defined,  $S'$  is imaginary, but we shall be able to effect a similar transformation on  $S'$  and get a real surface  $S''$ .

We have seen that, if  $\omega_1$  and  $\sigma$  be replaced by  $i\pi - \bar{\omega}_1$  and  $\pi - \bar{\sigma}$ , equations (18) are satisfied. Moreover, it can be shown that, if equations (55) are satisfied by

$$\sigma, \quad \omega_1, \quad p, \quad \rho, \quad r, \quad (57)$$

these equations are satisfied also by

$$\pi - \bar{\sigma}, \quad i\pi - \bar{\omega}_1, \quad -\bar{p}, \quad -\bar{\rho}, \quad \bar{r}, \quad (58)$$

where the bar indicates the conjugate imaginary function.

The successive application of these transformations upon  $S$  gives a surface  $S''$ , whose coordinates are of the form

$$\left. \begin{aligned} x'' = x - [i(p \sinh \omega_1 - \bar{p} \sinh \bar{\omega}_1) + (\rho \cosh \omega_1 + \bar{\rho} \cosh \bar{\omega}_1) \\ + (r \cos \sigma \cosh \omega_1 + \bar{r} \cos \bar{\sigma} \cosh \bar{\omega}_1)] X_1 \\ + [(p \cosh \omega_1 + \bar{p} \cosh \bar{\omega}_1) - i(\rho \sinh \omega_1 - \bar{\rho} \sinh \bar{\omega}_1) \\ - i(r \cos \sigma \sinh \omega_1 - \bar{r} \cos \bar{\sigma} \sinh \bar{\omega}_1)] X_2 \\ + (r \sin \sigma + \bar{r} \sin \bar{\sigma}) X. \end{aligned} \right\} \quad (59)$$

Hence the surface  $S''$  is real.

Among all the surfaces of Bonnet with a given spherical representation the origin itself may be counted. In this case we associate with it the trihedron, with vertex at the origin, rotating in such a way that its axes are parallel to the corresponding axes of the trihedron associated with a surface of Bonnet having the given spherical representation. Hence, if we put  $x, y, z$ , equal to zero in (59), these equations define all the real surfaces with a given spherical representation, when  $p, \rho, r, \sigma, \omega$  are given all the sets of values which satisfy (18) and (55); now  $E$  and  $G$  in (56) are zero also.

Since

$$\frac{\partial x'}{\partial u} = \sqrt{E'} X_1, \quad \frac{\partial x'}{\partial v} = \sqrt{G'} X_2; \quad \frac{\partial \bar{x}'}{\partial u} = \sqrt{\bar{E}'} X_1, \quad \frac{\partial \bar{x}'}{\partial v} = \sqrt{\bar{G}'} X_2,$$

where the bar indicates the conjugate function, for the surface defined by (59) (with  $x = y = z = 0$ ) we have

$$\sqrt{E''} = \sqrt{E'} + \sqrt{\bar{E}'}, \quad \sqrt{G''} = \sqrt{G'} + \sqrt{\bar{G}'}. \quad (60)$$

We consider several particular cases.

### § 7. *Particular Surfaces of Bonnet.*

Let  $\sigma = \frac{\pi}{2}$ ; from (34) it follows that  $\sigma_2 = \frac{\pi}{2}$  also. Now equations (55) reduce to

$$\left. \begin{aligned} \cosh \omega_1 \frac{\partial p}{\partial u} - i \sinh \omega_1 \frac{\partial \rho}{\partial u} - (p \sinh \omega_1 - i \rho \cosh \omega_1) \sinh \omega \cosh \omega_1 &= 0, \\ i \sinh \omega_1 \frac{\partial p}{\partial v} + \cosh \omega_1 \frac{\partial \rho}{\partial v} + (p \cosh \omega_1 - i \rho \sinh \omega_1) \cosh \omega \sinh \omega_1 &= 0, \\ \frac{\partial r}{\partial u} + (i p \sinh \omega_1 + \rho \cosh \omega_1) \cosh \omega &= 0, \\ \frac{\partial r}{\partial v} - (p \cosh \omega_1 - i \rho \sinh \omega_1) \sinh \omega &= 0; \end{aligned} \right\} \quad (61)$$

and by means of these equations the expressions (56) are reducible to

$$\left. \begin{aligned} \sqrt{E} &= -\frac{1}{\cosh \omega_1} \frac{\partial \rho}{\partial u} + i p \sinh \omega + r \cosh \omega, \\ \sqrt{G} &= \frac{i}{\sinh \omega_1} \frac{\partial \rho}{\partial v} - p \cosh \omega + i r \sinh \omega; \end{aligned} \right\} (62)$$

the accents have been removed.

Since equations (18) become

$$\frac{\partial \omega_1}{\partial u} + i \frac{\partial \omega}{\partial v} = -\sinh \omega \cosh \omega_1, \quad i \frac{\partial \omega_1}{\partial v} + \frac{\partial \omega}{\partial u} = \cosh \omega \sinh \omega_1, \quad (63)$$

three functions  $\alpha, \beta, \gamma$  may be defined in the following way:

$$\left. \begin{aligned} d\alpha &= \sinh \omega \sinh \omega_1 du + i \cosh \omega \cosh \omega_1 dv, \\ d\beta &= -ie^{-\alpha} [\sinh \omega \cosh \omega_1 du + i \sinh \omega_1 \cosh \omega dv], \\ d\gamma &= -ie^{\alpha} [\cosh \omega \sinh \omega_1 du + i \cosh \omega_1 \sinh \omega dv]. \end{aligned} \right\} (64)$$

If we put

$$\rho = c, \quad (65)$$

where  $c$  is a constant, the most general solution of equations (61) is

$$p = e^{\alpha} (\beta c + h), \quad r = \gamma (c \beta + h) - c \tau, \quad (66)$$

where  $h$  is an arbitrary constant and  $\tau$  is given by

$$\left. \begin{aligned} \frac{\partial \tau}{\partial u} &= (-ie^{-\alpha} \gamma \sinh \omega + \cosh \omega) \cosh \omega_1, \\ \frac{\partial \tau}{\partial v} &= (e^{-\alpha} \gamma \cosh \omega + i \sinh \omega) \sinh \omega_1. \end{aligned} \right\} (67)$$

We have neglected an additive constant for  $r$ , since it only tends to replace the surface now defined by surfaces parallel to it.

When  $c$  and  $h$  in (66) are real, all the surfaces of Bonnet defined by (59), with  $x = 0$  and  $p, q, r$  given by (65) and (66), are evidently homothetic to the surfaces which are the loci of the points dividing in constant ratios the joins of corresponding points on the two surfaces for which

$$c = 0, \quad h = 1; \quad c = 1, \quad h = 0.*$$

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\* cf. Surfaces Analogous to Surfaces of Bianchi, l. c. p. 121.

When  $\rho$  is not constant, the first two of equations (55) may be written

$$\left. \begin{aligned} \frac{\partial}{\partial u} e^{-\alpha} p &= i e^{-\alpha} \tanh \omega_1 \frac{\partial \rho}{\partial u} + \rho \frac{\partial \beta}{\partial u}, \\ \frac{\partial}{\partial v} e^{-\alpha} p &= i e^{-\alpha} \coth \omega_1 \frac{\partial \rho}{\partial v} + \rho \frac{\partial \beta}{\partial v}. \end{aligned} \right\} (68)$$

If  $p$  be eliminated from these equations by differentiating with respect to  $v$  and  $u$  respectively, it is found that  $\rho$  must satisfy the equation

$$\frac{\partial^2 \rho}{\partial u \partial v} - \frac{\partial}{\partial v} \log \cosh \omega_1 \frac{\partial \rho}{\partial u} - \frac{\partial}{\partial u} \log \sinh \omega_1 \frac{\partial \rho}{\partial v} = 0.$$

But this equation is satisfied by the function expressing the distance from the origin to the plane tangent to any surface of Bonnet whose spherical representation is given by (21). Hence if we know a solution  $\omega_1$  of equations (63) and also a surface with the representation (21), we can find by quadratures a surface with the representation (2).

It is easy to furnish an illustration of this remark. Corresponding to equations (63) for the representation (2), we have for the representation (21)

$$\frac{\partial \omega_3}{\partial u} + i \frac{\partial \omega_1}{\partial v} = -\sinh \omega_1 \cosh \omega_3, \quad i \frac{\partial \omega_3}{\partial v} + \frac{\partial \omega_1}{\partial u} = \cosh \omega_1 \sinh \omega_3.$$

A particular solution of these equations is

$$\omega_3 = \omega + i\pi.$$

Referring to (64), (65) and (66), we see that a surface with the representation (21) is defined by

$$x_1 = e^{-\alpha} (i \sinh \omega X'_1 - \cosh \omega X'_2) - \beta X', \quad (69)$$

and similar equations for  $y_1, z_1$ ;  $X'_1, X'_2, X'$  being the direction-cosines of the tangents to the parametric curves on the representation (21) and of the radius to the point on the latter with respect to the fixed  $x$ -axis. Now the distance of the tangent plane from the origin is  $-\beta$ . If this be substituted in (68), we have for the functions  $p$  and  $\rho$  determining a surface with the representation (2),

$$p = \frac{1}{2} \{e^{-\alpha} - e^{\alpha} (\beta^2 + k)\}, \quad \rho = -\beta,$$

where  $k$  is an arbitrary constant, and  $r$  is given by quadratures from (55). This case follows from (66) by taking  $h = 1, c = 0$  to determine (69); it is evident

that other solutions can be found by quadratures, when these constants are given other values.

In a similar manner we can find a large number of surfaces of Bonnet by methods analogous to those which we have used in getting the surfaces analogous to surfaces of Bianchi.\*

§ 8. *Determination of the Surface of Bonnet Applicable to a Given Surface of Bonnet.*

We have seen in § 5 that if  $\sigma'$  is defined by (44) the function  $\omega_1$  gives a Bäcklund transformation of a surface with the spherical representation (1). We will now use this fact to obtain a general method of determining surfaces with this representation similar to that established in § 6. Instead of starting with a surface having this representation we take the origin and associate with it a trihedron whose axes are parallel to the axes of the trihedron associated with the spherical surface having this representation.

If we denote by  $p'$ ,  $\rho'$ ,  $r'$  the functions analogous to  $p$ ,  $q$ ,  $r$ , as defined in § 6, the coordinates with respect to the fundamental trihedron of a point on a surface of the group can be written, in consequence of (44),

$$\begin{aligned} -[p' \cosh \omega_1 - i(\rho' + r' \sec \sigma) \sinh \omega_1], \quad -[i p' \sinh \omega_1 + (\rho' + r' \sec \sigma) \cosh \omega_1], \\ i r' \tan \sigma. \end{aligned}$$

Expressions for the projections upon the axes of a displacement upon the surface are similar in form to (54); from these it is found that the necessary and sufficient condition that the surface be a surface of Bonnet, with the given spherical representation of its lines of curvature, is that  $p'$ ,  $q'$ ,  $r'$  satisfy the conditions

$$\left. \begin{aligned} & \left( i \sinh \omega_1 \frac{\partial p'}{\partial u} + \cosh \omega_1 \frac{\partial \rho'}{\partial u} \right) \tan \sigma + i \sinh \omega_1 \cosh \omega \\ & \quad [p' \cosh \omega_1 - i(\rho' + r' \sec \sigma) \sinh \omega_1] = 0, \\ & \left( i \cosh \omega_1 \frac{\partial p'}{\partial v} + \sinh \omega_1 \frac{\partial \rho'}{\partial v} \right) \tan \sigma - \cosh \omega_1 \sinh \omega \\ & \quad [p' \sinh \omega_1 - i(\rho' + r' \sec \sigma) \cosh \omega_1] = 0, \\ & \tan \sigma \frac{\partial r'}{\partial u} = + [i p' \cosh \omega_1 + (\rho' + r' \sec \sigma) \sinh \omega_1] \sinh \omega, \\ & \tan \sigma \frac{\partial r'}{\partial v} = - [p' \sinh \omega_1 - (\rho' + r' \sec \sigma) i \cosh \omega_1] \cosh \omega. \end{aligned} \right\} (70)$$

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\* I. c. pp. 125-134.

The coefficients of the linear element of the surface are given by

$$\left. \begin{aligned} \sqrt{E_1} &= -\cosh \omega_1 \frac{\partial p'}{\partial u} + i \sinh \omega_1 \frac{\partial \rho'}{\partial u} - i (\rho' \sec \sigma + r') \sinh \omega \cot \sigma \\ &\quad - \sinh \omega_1 \cosh \omega \cot \sigma [p' \sinh \omega_1 - (\rho' + r' \sec \sigma) i \cosh \omega_1], \\ \sqrt{G_1} &= -i \sinh \omega_1 \frac{\partial p'}{\partial v} - \cosh \omega_1 \frac{\partial \rho'}{\partial v} - i (\rho' \sec \sigma + r') \cosh \omega \cot \sigma \\ &\quad + \cosh \omega_1 \sinh \omega \cot \sigma [p' \cosh \omega_1 - i (\rho' + r' \sec \sigma) \sinh \omega_1]. \end{aligned} \right\} (71)$$

Suppose now that we have given a surface,  $S$ , of Bonnet with the representation (2) and a solution  $\omega_1$  of equations (18). For  $S$  the functions  $p, \rho, r$  are known. From the equations

$$\sqrt{E_1} = \sqrt{E}, \quad \sqrt{G_1} = \sqrt{G},$$

when substitution has been made from (56)\* and (71), and the first two of equations (70) we get  $\frac{\partial p'}{\partial u}, \frac{\partial p'}{\partial v}, \frac{\partial \rho'}{\partial u}, \frac{\partial \rho'}{\partial v}$  in terms of known quantities. The conditions of integrability of these expressions and the last two of (70) are reducible to three linear equations in  $p', \rho', r'$ . Thus we find by algebraic processes  $p', q', r'$ , determining the unique surface  $S_1$  applicable to a given surface  $S$  with correspondence of lines of curvature; it has been shown that there always is a surface  $S_1$  of the kind sought.

If  $\sigma$  is such that condition (52) is satisfied, we can get at once another pair of applicable real surfaces of Bonnet as shown in § 5.

PRINCETON, *January*, 1906.

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\* Here  $\sqrt{E}$  is  $\sqrt{E'}$  of (56) and  $E$  of the latter is zero.